Recursive Derivation of the Euler's Continued Fraction

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In a previous essay[3], using a constructive process, I could obtain an inspiring pattern that led to the expression of Euler's[4] Continued Fraction (CF):

$$a_0+a_0a_1+\ldots+a_0a_1a_2\cdots a_n=\cfrac{a_0}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2-\cfrac{\ddots}{\ddots\cfrac{a_{n-1}}{1+a_{n-1}-\cfrac{a_n}{1+a_n}}}}}} \tag{1}$$

As in the previous essay, I am continuing trying to figure out what was Euler's intention when he conceived his continued fraction, and how he obtained it. I am not claiming that this was never done before; I have not thoroughly searched the literature. I am just trying to figure out how the original expression was obtained.

In that essay, the constructing process consisted of obtaining closed-form expressions for the finite sum of products in the left side of Eq. 1 for $n=1$ to $n=3$, namely

$$a_0+a_0a_1=\cfrac{a_0}{1-\cfrac{a_1}{1+a_1}}$$

$$a_0+a_0a_1+a_0a_1a_2=\cfrac{a_o}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2}}}$$

$$a_0+a_0a_1+a_0a_1a_2+a_0a_1a_2a_3=\cfrac{a_1}{1-\cfrac{a_1}{1+a_1-\cfrac{a_2}{1+a_2-\cfrac{a_3}{1+a_3}}}}$$

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3 Sampaio Jr., J H B S – Intuitive Derivation of Euler's Continued Fraction – Dec 2023 – ResearchGate preprint DOI 10.13140/RG.2.2.30718.77120 - LicenseCC BY-NC-SA 4.0

4 Leonard Euler, 15 Apr 1707 – 18 Sep 1783 (https://en.wikipedia.org/wiki/Leonhard_Euler)

and noting that the pattern for increasing terms would indicate the general expression in Eq. 1. If the Euler's Continued Fraction was unknown, by reaching the last expression above someone would be induced (maybe Euler himself) to try to prove it using mathematical induction. This is what appears in a large number of sources e.g., math books, articles, and online sites.[5]

After I've written (and published) that essay, I realized that one could directly deduct Euler's CF by recursion, i.e., by finding a recursive term, and without the need of mathematical induction. All one needs is to use a general procedure similar to that in the essay. This would be a direct derivation.

Thus, we start with the sum of products Euler would have, somehow, in hand

$$a_0+a_0a_1+\ldots+a_0a_1a_2\cdots a_n$$

and express it in the following way:

$$a_0+a_0a_1+\ldots+a_0a_1a_2\cdots a_n=a_0\left(1+a_1+\ldots+a_1a_2\cdots a_n\right)$$

$$\cdots=\frac{a_0}{\dfrac{1}{1+a_1+\ldots+a_1a_2\cdots a_n}}$$

Using the same directive of avoiding ambiguity cited in the essay, we proceed by adding and subtracting the sum $a_1+a_1a_2+\ldots+a_1a_2\cdots a_n$ to the numerator of the lower division to obtain

$$\cdots=\frac{a_0}{\dfrac{1+\left(a_1+a_1a_2+\ldots+a_1a_2\cdots a_n\right)-\left(a_1+a_1a_2+\ldots+a_1a_2\cdots a_n\right)}{1+a_1+\ldots+a_1a_2\cdots a_n}}$$

and factor *1* out of the fraction. Then, we factor a_1 out of the numerator of the lower fraction:

$$\cdots=\frac{a_0}{1-\dfrac{a_1+a_1a_2+\ldots+a_1a_2\cdots a_n}{1+a_1+\ldots+a_1a_2\cdots a_n}}=\frac{a_0}{1-\dfrac{a_1\left(1+a_2+\ldots+a_2a_3\cdots a_n\right)}{1+a_1+\ldots+a_1a_2\cdots a_n}}$$

Now we reorganize the fraction in the denominator and get:

$$a_0+a_0a_1+\ldots+a_0a_1a_2\cdots a_n=\frac{a_0}{1-\dfrac{a_1}{\dfrac{1+a_1+\ldots+a_1a_2\cdots a_n}{1+a_2+\ldots+a_2a_3\cdots a_n}}} \tag{2}$$

Now, we will focus in the lower rightmost fraction in the previous expression (in red):

$$\frac{1+a_1+\ldots+a_1a_2\cdots a_n}{1+a_2+\ldots+a_2a_3\cdots a_n}$$

This fraction can be expressed in the following particular indexed form:

$$\frac{1+a_k+\ldots+a_ka_{k+1}\cdots a_n}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}\quad,\qquad (k=1) \tag{3}$$

Note that the numerator contains $n-k+2$ terms starting with $1+a_k$ and the denominator contains $n-k+1$ terms starting with $1+a_{k+1}$. Note also that the denominator *lacks* a_k.

5 For example, https://en.wikipedia.org/wiki/Euler's_continued_fraction_formula

We now perform the following algebraic manipulation of the generic[6] indexed fraction in Eq. 3

$$\frac{1+a_k+\ldots+a_k a_{k+1}\cdots a_n}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n} \tag{4}$$

Firstly, we add and subtract $a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n$ to the numerator, to obtain:

$$\cdots=\frac{1+\color{red}{a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n}+a_k+\ldots+a_k a_{k+1}\cdots a_n\color{red}{-\left(a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n\right)}}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}$$

The terms added and subtracted are shown in red. Now we factor out a_k from the $n-k+1$ central terms of the numerator starting in $a_k+a_k a_{k+1}$ (in black), to get

$$\cdots=\frac{1+a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n+a_k\left(1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n\right)-\left(a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n\right)}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}$$

Then we factor out $1+a_k$ from the fraction (noting the exact division by the denominator), and obtain

$$\cdots=1+a_k-\frac{a_{k+1}+a_{k+1}a_{k+2}+\ldots+a_{k+1}a_{k+2}\cdots a_n}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}$$

The final step here is to factor out a_{k+1} from the numerator of the previous expression and transfer the remaining factor to the denominator:

$$\cdots=1+a_k-\frac{a_{k+1}\left(1+a_{k+2}+\ldots+a_{k+2}a_{k+3}\cdots a_n\right)}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}=1+a_k-\frac{a_{k+1}}{\frac{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}{1+a_{k+2}+\ldots+a_{k+2}a_{k+3}\cdots a_n}}$$

Then we reach the following recursive identity:

$$\frac{1+a_k+\ldots+a_k a_{k+1}\cdots a_n}{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}=1+a_k-\frac{a_{k+1}}{\frac{1+a_{k+1}+\ldots+a_{k+1}a_{k+2}\cdots a_n}{1+a_{k+2}+\ldots+a_{k+2}a_{k+3}\cdots a_n}} \tag{5}$$

We have here established the recursive decomposition of the generic indexed fraction of order k in Eq. 3 into the difference between $1+a_k$ and a term that does not contain a_k. This term is a_{k+1} divided by the generic indexed fraction of order $k+1$ in Eq. 3. In other words we factor $1+a_k$ out of the fraction in Eq. 4, and increased the index towards n by *1* in the remaining factor. This process repeats over and over until we extract the term $1-a_{n-1}$, which is the base case of the recursion:[7]

$$\frac{1+a_{n-1}+a_{n-1}a_n}{1+a_n}=1+a_{n-1}-\frac{a_n}{\frac{1+a_n}{1}}$$

At this occurrence, the recursion stops because there is no more terms to factor out, and the expression of Euler's Continued Fractions (Eq. 1) results immediately from exhaustive substitution of Eq. 5 into Eq. 2. There is no need to prove it using mathematical induction.

The Euler's CF can be extended to a infinite sum of products of the form $a_0+a_0a_1+\ldots+a_0a_1a_2\cdots$, with the fractions also extending infinitely.

6 For no particular value of *k*

7 It is a formality and a tradition, in certain texts, to keep explicitly the denominator *1* in the fraction in the lower denominator, although being redundant.